[SQUEAKING]

[RUSTLING]

[CLICKING]

PETER SHOR: OK. recall. So of course, I'm going to go over a few things from last time. Suppose we have a sub i things of size i. So these could be-- well, I mean, they could be Dyck walks, in which case, a sub i is the Catalan number. And the generating function is summation a sub i x.

OK. The notes generally use n for this, so I'm going to try to do that, but I'm probably going to not be consistent. The generating function is a of x. Summation, a sub n x to the n. n equals-- well, let's say 0 to infinity. But, I mean, maybe we want to start counting at 1 in certain cases.

OK. And suppose we have two classes of things, a sub i and b sub i. Make that a sub n and b sub n, and c sub n equals a sub n plus b sub n, where c is the union of A and B. Or the disjoint union. Then c of x is equal to A of x plus B of x.

And suppose c equals the direct product of A and B, where the size of c-- so c is a, comma, b, something from a and something from b. So the size of c is equal to the size of a plus size of b. Then c sub n equals summation i equals 0 to n, a sub i, b sub n minus i. Because if you want two things to sum to n, one should be i and the other should be n minus i.

And c of x is equal to A of x times B of x. And there's one last way of manipulating generating functions that I did not discuss last time. Actually, there's more than that. I mean, you can differentiate them and integrate them. But I gave you an example of that last time. But let's let sequence of A be the set of finite sequences of things in A.

So let's do an example that will illustrate this. Suppose A is equal to 0 and 1. Sequence of A is equal to binary strings. Oops. 0, 0 is in A cubed, for example. So what is the generating function for sequence of A?

I want to claim S of x is equal to sum from i equals 0, n equals 0 to infinity. A. OK. Let's write it out first. Is equal to 1 plus A of x plus A of x squared plus A of x cubed. So this is the empty sequence. This is 0, 1. This would be 0, 0, 0, 1, 1, 0, 1, 1 for binary sequences, et cetera.

And A of x equals 2x for binary sequences, because here, there are two binary sequences of length 1 and x means length 1 here, and 2 means there are two of them. So S of x is equal to 1 plus 2x plus 4x squared plus 8x cubed plus dot, dot, dot, which you probably all recognize as 1 over 1 minus 2x. We can go up here and say this equals 1 over 1 minus A of x.

I guess this calculation is not very impressive for binary strings because everybody knows how to come up with this in much easier ways. Next what we're going to do is we're going to do Fibonacci numbers. And Fibonacci numbers are a little bit harder, and we will see that the generating function actually gives you interesting things for Fibonacci numbers.

OK. So Let's start with another question. Suppose we want to send information over a-- we want to send information over a telegraph type channel where we can send a dot and a dash. And the question is, how much information or how many messages can we send?

Can we send n time n. And I forgot to tell you, a dot has length 1 and a dash has length 2. OK. Let's start calculating it. With time 1, All we can do is send a dot. So that's not really a message. I mean, if you can only send one message, it contains no information. At time 2, we could either send a dash or two dots.

At time three, so that's two possibilities. This is one possibility. In time 3, we can either send a dash and a dot, or a dot and a dash, or three dots. That's three. In time 4, we send dot, dot, dot, or dash, dash, dash, dot, dot, dot, dot, dash, and dot, dot, dot, dot. So that's five messages. Does anybody recognize this sequence?

OK. Let's do the next one. Well, there are two dashes and a dot, or dot and two dashes, or dash, dot, dash. And we could dash and three dots, dot, dash, dot, dot dot, dot, dot, dash, dot, dot, dot, dot, dash, and dot, dot, dot, dot, dot, dot. So there's eight of these things.

OK. Do people recognize the sequence now? What is it? Fibonacci numbers. Yeah. So we get the Fibonacci numbers. And actually, there's a very easy way to prove that we get the Fibonacci numbers. I want to claim f sub n equals f sub n minus 1 plus f sub n minus 2. So this is if something ends with a dot. So sequence of length n is either going to end with a dot or a dash.

If it ends with a dot, then there are fn minus 1 ways of filling the first n minus 1 times units. If it ends with a dash, there are fn minus 2 ways of filling the first n minus 2 times. So you get this recurrence for the Fibonacci numbers, which I'm sure everybody already knows about. So what we're going to do is we're going to ask, how can we use generating functions to find a formula for the i-th Fibonacci number?

And I wanted to say something. Back when our lecture notes were first written, I used f1 equals 1, f0. There's one way of sending message of like 0, which is the empty message. So f0 equals 1, f1 equals 1. f2 equals 2, f3 equals 3, f4 equals 5, et cetera. So since then, Wikipedia has installed itself as the default arbiter of definitions, et cetera.

And today, Wikipedia says f0 equals 0, f1 equals 1, f2 equals 1, f3 equals 2, and everything is shifted by one. And I have not had time to rewrite the notes. So the notes disagree with Wikipedia's definition of Fibonacci numbers, and today, Wikipedia basically is the default definition for everything, even of some references disagree. OK.

And it's much easier to do this calculation. I think it's much easier to do this calculation. It's a little easier to do this calculation. And we use the alternate definition rather than Wikipedia's definition. But let's write down the generating function for f of x of x equals summation f of x summation the i-th the induction number times x to the-- n-th Fibonacci number times x to the n, which is equal to 1 plus x plus x squared plus 3x cubed plus 5x to the fourth, plus 8x to the fifth, plus 13x to the sixth, dot, dot, dot.

OK. So how do we compute the value? How do we find a closed form for f of x? Well, let's take f of x equals x plus x squared plus 2x cubed plus 3x cubed plus 5x to the third, and x squared f of x equals x squared plus x cubed plus 2x fourth plus 3x to the fourth plus dot, dot, dot. OK. So now we have these three formulas. Can anybody tell me what I should do with them?

AUDIENCE: Subtract them.

PETER SHOR: Subtract them. Very good f of x minus x. f of x minus x squared. f of x equals. Can anybody tell me what this is? One. Exactly. So it's one, because anything after the first term, we have f sub i, f sub i minus 1, and f sub i minus 2, and we take this and subtract that and subtract that. We get zero. The first term doesn't quite work like that. We take 1 minus 0 minus 0 and we get 1.

And of course, this gives you the equation f of x 1 minus x minus x squared equals 1 or f of x equals 1 over 1 minus x minus x squared. OK. So how do we find a formula for f of x from this? How many of you remember from calculus the method of partial fractions? So some of you do. How many have never seen the method of partial fractions? OK, so most of you remember it.

I mean, I guess there's some question as to when we have Mathematica, does anybody need to remember how to integrate expressions? But I guess they haven't taken away all the integration methods from calculus yet. OK. So what do we do? We use the method of partial fractions. We say 1 minus x minus x squared is equal to 1 minus phi plus 1 minus phi minus.

And phi plus, phi minus are the roots of-- well, what we're going to do is we're going to reverse the order of these coefficients and say y squared minus y minus 1. And ah. I forgot to put the x's in this equation. Yeah. Maybe I should explain why phi plus and phi minus are roots of y squared minus 1 minus 1.

So y squared minus y minus 1 equals y minus y plus y minus y minus. And let's let x equal 1 over y, 1 over x2 squared minus 1 over x minus 1. Wait. This should be y squared plus y minus 1. y squared plus y minus 1. And now 1 over x squared plus 1 over x minus 1 equals 1 over x minus phi plus 1 over x minus phi minus.

I think I've done something wrong. Let me think about this. So we have 1 minus x minus x squared equals k. And now we want y squared plus y. This should be if y is 1 over x, then this corresponds to the 1. This corresponds to y. This corresponds to y squared. But this really should be minus y, minus y squared plus 1 equals that. And now we have-- that doesn't work. OK.

Oh, yeah. Yeah, this is right. OK. Good. And multiplying both sides of this by x squared, we get 1 minus phi plus x, 1 minus phi minus x on the right side. And we should get 1 minus x minus x squared. So this should be a minus, and this should be a minus, and this should be a minus.

Yeah. OK. Good. So the roots of y squared. Roots are negative 1, 1 plus or minus the square root of 5 over 2 by the quadratic formula. And these are phi plus and phi minus. And phi plus will have the plus sign and phi minus will have a minus sign. Wait, no. I wanted-- I'm sorry. I want to keep this equation. OK.

AUDIENCE: Sorry. How did it get there, root of y squared minus 1?

PETER SHOR: Oh. So what I did was I replaced y by 1 over x and then multiplied by y squared. And I clearly did this too fast because I confused myself, so let's do it. And I confused you too by going too fast, so let's go over more slowly. OK. Let's let y equal 1 over x. x. So this is 1 minus 1 over phi minus 1 over phi squared is equal to 1 minus phi plus over y. 1 minus phi minus over y.

OK. And now multiply both sides by y squared. So this is y squared minus y minus 1 equals y minus phi plus y minus phi minus. So phi plus is the largest root of y squared minus y minus 1, and y minus is the smaller root of y squared minus y minus 1. So we have phi plus is equal to 1 plus root phi over 2, and phi minus equals 1 minus phi over 2. OK.

OK. So what do we have? We have f of x equals 1 over 1 minus 1, x minus x squared equals 1 over 1 minus phi plus x, 1 minus phi minus x. And by partial fractions, we know this is equal to a over 1 minus phi plus x plus b over 1 minus phi minus x.

OK. So what do we do with this? Well, let's try to solve this equation. So this is equal to a times 1 minus phi minus x plus b times 1 minus phi plus x over 1 minus x minus x squared. Because this times this is 1 minus x minus x squared. So now we have this is equal to this, so this gives us two equations. We have a plus b equals 1, and a phi minus x plus b phi a-- yeah. a phi minus x plus b phi plus x equals 0x.

OK, so we have two equations. We have two unknowns, a and b, and you know how to solve them. And I am not going to do it because it's a lot of algebra and I quite possibly would screw up somewhere along the way. And you know how to do it anyway. So the answer comes out to a equals 1 over root phi times 1 plus root phi over 2, and b equals 1 over root phi times 1 minus root phi 2. So this is phi plus and this is phi minus.

So the formula for f. Well, f of x equals a over 1 minus y plus x plus b over 1 minus y minus x is equal to a times 1 plus phi plus x plus phi plus squared x squared plus phi plus cubed x cubed. And plus b times 1 plus phi minus x plus phi minus x squared x squared plus phi minus cubed x cubed plus dot, dot, dot.

And of course, f of x was equal to summation x sub i, x sub n, x to the n. So to find the n-th Fibonacci number, we need to find the n-th term in the sequence. So we get f sub n is equal to a phi plus to the n plus b phi minus to the n. And let's go back and see what a and b were. a was 1 over root phi times phi plus and b was 1 over root phi times phi minus.

So this is equal to 1 over root phi phi plus to the n plus 1 plus phi minus n plus 1. Where phi plus is 1 over root phi plus 1 plus root phi over 2. And phi minus is 1 minus root phi over 2. So here, we have a term for the n-th Fibonacci, formula for the n-th Fibonacci number. Oh, wow. OK.

So how fast does the n-th Fibonacci number grow? How many messages can we send in time n with dots and dashes? Well, it's just this formula. It's 1 over root phi, phi plus to the n plus 1 phi minus to the n plus 1.

OK. Now I want to say that phi plus is bigger than 1, and phi minus is less than 1. So this is just very approximate and very close to 1 over root phi, phi plus to the n plus 1. So the information transmission rate is log base 2 phi plus to the n plus 1 over root phi, which is approximately-- well, we want to divide this by the time per time unit.

Let's say information rate per time unit is log 2 phi plus 2n plus 1 root phi over n, which is approximately-- so this n plus 1 comes close to canceling this n. Root phi is just a constant factor, a constant addition factor which is negligible. So it's log 2 of phi plus.

So if you have a channel that uses dots and dashes to transmit information, this is how much you can transmit in time n. And we're actually going to be going back to this later in the course and talking about information theory and how much information you can transmit in general. OK. So let's see. I'm trying to think if I have time to do this next bit. I think I do. So let's answer another question.

Oh, wait. Before we answer another question, I want to go back and say, we actually could have gotten this formula. Where is it? f of x equals 1 over 1 minus x minus x squared, much more easily. So how could we have gotten the generating function for this dots and dashes problem much more easily? Anybody?

OK. So yeah, it's not completely obvious. So f of x is number of sequences of dot and dash. So the generating function at x equals 1, I'm sorry x plus x squared for dot, dash, because dots have time 1, which is x, dashes have time 2, which corresponds to x squared. And we just add them because the generating function when you have two disjoint things, you take the sum.

Does the sequence of dots and dashes is equal to. Well, we use this formula up here, 1 over 1 minus A of x is 1 over 1 minus x minus x squared. And we just saved ourselves a lot of effort by using that formula rather than deriving it through writing out the recurrence and adding the equations up and seeing what the result was. So maybe we should try applying this to a different question.

It's a tiling question, which you're not getting on your homework. I think there's a tiling question on your homework. The generating function or number of tilings of 2 by n strip with two kinds of tiles. A triangular tile and a zigzag tile.

AUDIENCE: Can you--

PETER SHOR: Say something?

AUDIENCE: Can you rotate or flip?

PETER SHOR: Yeah, you're allowed to rotate or flip. Otherwise you couldn't tile the strip without flipping this one. So for three, you get-- and it's for six, you get-- no, let's do five first. For five, you get. Yeah, that's five. 1 plus 2 plus 2. And it's flipped. I won't draw it.

And for six what you can get is you can get. I'm doing this wrong. Six, you can get, and now you can join another unit of six. OK. So how would you find the generating function for this?

Well, let's use this sequence formula, Ax is equal to. Well, there's two ways of tiling a 3 by 2 strip. So that's x cubed times 2 plus-- well, there are two ways of tiling of a five strip. So that's 2x to the fifth. There are two ways of tiling a seven strip. I mean you just stick two z tiles instead of one z tile.

So that's plus 2x cubed plus dot, dot, dot. And that is equal to 2x cubed over 1 minus x squared. So g of x is equal to 1 over 1 minus 2x cubed over 1 minus x squared is equal to 1 over-- I won't say 1 minus x squared minus 2x cubed. And I want to put in 1 minus x squared on top. OK.

OK. And you can figure out what the recurrence for this is, which is really weird. But maybe I shouldn't say weird. Which is really quite non-intuitive. But we had the recurrence fn equals fn minus 1 plus fn minus 2. That came up to 1 minus x minus x squared. So this is going to be fn is equal to fn minus 2 minus 2 fn minus 3.

And the top says what the initial conditions are. f0. Wait, this should be a plus. f0 equals 1 and f2 equals minus 1. And I'm not going to have time to explain why going from here to here makes this, but let's see what sequence we get. And I should not say this is f2 equals 93. Because, I mean, 1 minus x-- what was that? 1 over 1 minus x squared minus 2x cubed.

The initial condition is 2 equals 1, f of 0 equals 1, and f sub i equals f of i minus 2 plus 2 and minus 3. And x squared over 1 minus x squared minus 2x cubed is the same thing, except the initial condition is f of 2 equals minus 1 and the same recurrence. So we can look at the recurrence and see if we actually get what we wanted.

So 1. So this is 0, 1. 2. We get 1 here. 3, we get 1. We get 2 here. 4. We look back 2 and we get another 1. Is that right? Yeah, that's right. And now we subtract minus 1, 0, 1. For 5, we look back 2 and we get 2 and we add 2 times this. So this should be 4.

And now we subtract 1, 0, 1, 0, because subtracting-- shifting over 2 and subtracting is the same as multiplying by minus x squared. And we get this. Oh, this should be 2. 1, 0, 1, 2. And we get 1, 0, 0, 2, 0, 2. And indeed there were 2 for 5. There were 2 for-- there was 2 for 3, and there was 1 for length 0, which is just the empty tiling.

And the next one should give us-- the next element should really give us 4 if I've calculated it right. And it probably does. I'm not going to check it. So that is how you could do another tiling. OK, so now let's switch to a different question. Let's look at Pascal's triangle. OK.

OK. Pascal's triangle is 0, 0, 1 choose 0, 1 choose one, 2 choose 0, 2 choose 1, 2 choose 2, et cetera. And let's look at n numbers. 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1, 1, 5, 10, 5, 1, et cetera. So let's do this. Now suppose you're just looking at this and you start to sum and you ask, what happens if I sum, I guess, diagonals of it?

I mean, let's take 1 plus 1 plus 2. 1 plus 3 plus 1. That's 5. 1 plus 6 plus 5 plus 1. That's 13. And you happen to notice that these are Fibonacci numbers. I mean, 2 is the third Fibonacci number, 5 is the fifth of Fibonacci number, 13 is the seventh Fibonacci number, et cetera. OK.

So once you've noticed this, you can ask, how can I prove it? And actually, there are a whole bunch of ways to prove it. You can prove it by counting something two different ways. You can prove it by straightforward induction, and you can prove it using generating functions. And I'm going to prove it using generating functions. And I have to say, I think this proof is written up fairly badly in the notes. But it's in the notes. OK.

Proof using generating functions. OK. Well, first, let's compute the generating function for Pascal's triangle. Now, this is a two dimensional array of numbers so we're going to need two variables to do this. And we'll call this x to the 0, y to the 0. x to the y and y to the 0. x to the 0, y to the 1, et cetera. So what we get is we get x plus y choose x, x equals 1, 0 to infinity. Some y equals 0 to infinity. x plus y, which is x, times x to the--

OK. I am not looking at my notes because in my notes I'm sure I did not use x for two different things. So let's see. Oh. Yeah. Some a equals 0 to infinity, b equals 0 to infinity. a plus b, choose a. x to the a, y to the b. So this is the generating function for Pascal's triangle. Two dimensional generating functions, so you get two different variables.

So what is it? a plus b, which is x to the a, y to the b. OK. So what I want to do is I want to break this up into a equals 0 to n and b equals n minus a, and then-- so n equals 0 to infinity. So I'm going to break this up and just sum it along the rows first and then sum all the rows after that.

So single row is sum n choose a, a equals 0 to n, x to the a, y to the n minus a. Can someone tell me what this is?

AUDIENCE: [INAUDIBLE]

PETER SHOR: Yeah.

AUDIENCE: x plus y.

PETER SHOR: Right. This is x plus y to the n. Good. So sum n equals 0 to infinity, x plus y to the n is equal to 1 over 1 minus x minus y. OK. Well, what do we do with that? Well, what we're going to have to do is we're going to have to go back and look at the sum a little more carefully.

OK, so there are 1, 1, 1, 1, 2, 1, 1, 3, 3, 1, 1, 4, 6, 4, 1. Let's take 1, 5, 10, 10, 5, 1. And there's a 1 over here. So let's look at this diagonal. 1, 6, 5. That's 13. OK. So this turns out to be 3 choose 0 plus 3 plus 0 choose 0 times x cubed y to the 0.

Plus 2 plus 2 choose 2. x squared y squared plus 1 plus 4 choose 4. That's 5 choose 4. That's this. xy cubed plus 0 plus 6 choose 6, x to the 0, y to the. This is y4. This is y 6. So what we're doing is we're going along and we're subtracting 1 from m and adding 2 to n each time.

So I want to say this is summation i equals 0 to k. Let's call this n minus i plus 2i choose 2i x to the 2i, x to the n minus i, y to the 2i. OK. So what we want is we want to somehow group all these terms together in the generating function. Can anybody tell me how to do that?

OK. x equals z squared. And let's let y equal z. OK. So now what we're going to do is this becomes summation i equals 0 2n, n plus i choose 2i, x to the-- or z to the 2n minus 2i times z to the 2i. And this is z to the 2n.

So if we plug the generating function, if we plug x equals z squared and y equals z into the generating function for Pascal's triangle, the generating function we get will give us these, I guess diagonal sums. Is this clear? OK. Great. So what was the generating function for Pascal's triangle? It was 1 minus x minus y.

OK. So substitute x equals-- what was that? x equals z squared. xz squared, y equals z, and 1 over 1 minus x minus y. You get 1 over 1 minus z minus 1 over z squared minus z. And this is a generating function for the Fibonacci numbers. And you can go back and you can look at this.

And you can realize that our proof not only gives you the Fibonacci numbers for these diagonals, it also, if you look at the other diagonals, 2 plus 1 is 3. And I want to say 3 plus 4 plus 1 is 8. I mean, these diagonals starting on the left side of the triangle give you the odd numbered Fibonacci numbers, or the every other Fibonacci numbers. These diagonals starting on the right side give you the Fibonacci numbers between them.

OK. And that's actually all I wanted to say about generating functions today. We're going to be doing generating functions for Catalan numbers on Tuesday. But I want to hold that off because I want to do everything on the same day. So I thought maybe I should show you the proof of the same recurrence with counting something in two different ways.

So Fibonacci numbers. Number of length n. Sequences of dots and dashes where the dashes are length 2 and the dots are length 1. OK. OK. And if n equals 5, the number of sequences is 8. So we will be doing that one.

So let's group sequences by number of dots. OK. So we can have eight dots. Well, we can have zero. We can have one dot. Two dashes. We could have two dots. Well, we can't have two dots. We can have three dots and one dash, or we could have five dots and zero dashes. So if you have one dot and two dashes, how many ways are there of arranging these things?

AUDIENCE: Three.

PETER SHOR: Yeah. It's 1 plus 2 choose 2. If you have three dots and one dash, you get 3 plus 1 choose 1. And if you have five dots and zero dashes, you get 5 plus 0 choose 0. And I want to claim that this is essentially this sum. So summation. I'm going to say an plus i, i equals 0 to something choose. This is 3. This is 3 plus 1, plus 1 is 4. And this is 1. And that corresponds to this diagonal 3 plus 4 plus 1.

So this is a way of proving this formula by counting things in two different ways. You count the number of sequences of dots and dashes of length n. And first you do it by just saying that the Fibonacci number. The second way you do it is by grouping them by the number of dots and dashes in each one, and twice the number of dots plus the number of dashes is equal to the length of the sequence, and summing them up gets you these diagonals. OK, any questions about this?

AUDIENCE: Can you explain, what's the last thing that you connected it to? The last thing you said. You're connecting it to Pascal's triangle?

PETER SHOR: Yeah, I'm connecting it with Pascal's triangle. Actually, I probably should have. OK. I have a formula in here, I believe. OK. Summation m plus j choose 2j, j equals 0 2m.

Claim. This is the number of ways of arranging dots and dashes to have length. Let's see. So I'm going to call this m minus j plus 2j choose 2j, and to have length m. If we have m minus j dashes, 2j dots, you get length m.

Yeah. I mean, the dots are length 1, the dashes are length 2, so this is length 2m. And number of ways to arrange m minus j functions, 2j dots is m minus j plus 2j choose 2j. Which is m plus j choose 2j.

So sum m plus j choose 2j is a number of ways of breaking 2m. 2m. No. Of arranging dots and dashes to have length 2m total. So this is equal to Fibonacci number of length 2m, maybe plus or minus. I mean, or maybe this is 2m plus 1 or negative 1 or something. But anyway, this gives you the formula.

OK. So next time we're going to derive the formula for the Catalan number, which, I can't remember if I told you what it was already or not. I think I did. We derived it using Dyck walks. Next time, we will derive it using generating functions and a recurrence for the Catalan number. And that will, I think, finish our section on generating functions. And I think--